Vector identities (Griffiths)

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$
$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$
$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot\nabla f$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$
$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$
$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Vector identities (Jackson)

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$\int_V \nabla\cdot\mathbf{A}\,d^3x = \int_S \mathbf{A}\cdot d\mathbf{a} \qquad (1)$$
$$\int_V \nabla\psi\,d^3x = \int_S \psi\,d\mathbf{a}$$
$$\int_V \nabla\times\mathbf{A}\,d^3x = \int_S d\mathbf{a}\times\mathbf{A}$$
$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,d^3x = \int_S (\phi\nabla\psi)\cdot d\mathbf{a} \qquad (2)$$
$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{a} \qquad (3)$$
$$\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int_C \mathbf{A}\cdot d\mathbf{l} \qquad (4)$$
$$\int_S d\mathbf{a}\times\nabla\psi = \int_C \psi\,d\mathbf{l}$$

(1) divergence theorem; (2) Green's first identity; (3) Green's theorem; (4) Stokes's theorem

Spherical coordinates (Griffiths)

$$d\mathbf{l} = \hat{\mathbf{r}}\,dr + r\hat\theta\,d\theta + r\sin\theta\,\hat\phi\,d\phi$$
$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat\theta - \sin\phi\,\hat\phi$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat\theta + \cos\phi\,\hat\phi$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat\theta$$
$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat\theta = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\nabla t = \partial_r t\,\hat{\mathbf{r}} + \frac{1}{r}\partial_\theta t\,\hat\theta + \frac{1}{r\sin\theta}\partial_\phi t\,\hat\phi$$
$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\partial_\phi v_\phi$$
$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\hat{\mathbf{r}}[\partial_\theta(v\phi\sin\theta) - \partial_\phi v_\theta] + \frac{1}{r}\hat\theta\left[\frac{1}{\sin\theta}\partial_\phi v_r - \partial_r(r v_\phi)\right] + \frac{1}{r}\hat\phi[\partial_r(r v_\theta) - \partial_\theta v_r]$$
$$\nabla^2 t = \frac{1}{r^2}\partial_r(r^2\partial_r t) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta t) + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi}t$$

Cylindrical coordinates (Griffiths)

$$d\mathbf{l} = \hat{\mathbf{s}}\,ds + s\hat\phi\,d\phi + \hat{\mathbf{z}}\,dz$$
$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat\phi \qquad \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat\phi$$
$$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \qquad \hat\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\nabla t = \partial_s t\,\hat{\mathbf{s}} + \frac{1}{s}\partial_\phi t\,\hat\phi + \partial_z t\,\hat{\mathbf{z}}$$
$$\nabla\cdot\mathbf{v} = \frac{1}{s}\partial_s(s v_s) + \frac{1}{s}\partial_\phi v_\phi + \partial_z v_z$$
$$\nabla\times\mathbf{v} = \hat{\mathbf{s}}\left[\frac{1}{s}\partial_\phi v_z - \partial_z v_\phi\right] + \hat\phi[\partial_z v_s - \partial_s v_z] + \frac{1}{s}\hat{\mathbf{z}}[\partial_s(s v_\phi) - \partial_\phi v_s]$$
$$\nabla^2 t = \frac{1}{s}\partial_s(s\partial_s t) + \frac{1}{s^2}\partial_{\phi\phi}t + \partial_{zz}t$$

Useful variation

$$\delta(ds/d\theta) = \frac{(dx^i/d\theta)(d\delta x_i/d\theta)}{ds/d\theta} \quad (54.2)$$

Primed frame moves with velocity $v\hat{\mathbf{x}}$ relative to unprimed frame.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Transformation of the field (HW3, problem 2.1)

$$E'_x = E_x \quad E'_y = \gamma(E_y - \beta B_z) \quad E'_z = \gamma(E_z + \beta B_y)$$
$$B'_x = B_x \quad B'_y = \gamma(B_y + \beta E_z) \quad B'_z = \gamma(B_z - \beta E_y)$$

Basic objects of relativistic electrodynamics

$A^i = (\phi, \mathbf{A}) \quad A_i = (\phi, -\mathbf{A}) \quad F_{ij} = \partial_i A_j - \partial_j A_i$

$\mathbf{E} - \nabla\phi - \partial\mathbf{A}/\partial t \quad \mathbf{B} = \nabla\times\mathbf{A}$

$$F_{ij} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$
$$F^{ij} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}B_\gamma$ (p. 80) $\quad \epsilon^{0123} = +1 \quad \epsilon_{0123} = -1$

$F^{ij}F_{kl} \propto B^2 - E^2 \quad F^{ij}F^{kl}\epsilon_{ijkl} \propto \mathbf{E}\cdot\mathbf{B}$ (p. 85)

$F^{ij}F^{kl}\epsilon_{ijkl}$ is a boundary term (4-divergence)

Four-current (p. 96)

$$e\int u^i A_i\,ds = \frac{1}{c}\int A_i j^i\,d^4x$$
$$j^i(x^j) = \sum_A \left[c e_A \int u_A^i \delta^4(x^j - x_A^j(\tau))\,ds\right]$$

Four-current for EM field (p. 101): $j^i = (c\rho, \mathbf{j})$

Action for scalar field (5) and EM field (6)

$$S = -mc\int ds + \int \phi(x^i)\,ds \qquad (5)$$
$$S = -mc\int ds - \frac{e}{c}\int u^i A_i\,ds - \frac{1}{16\pi c}\int F^{ij}F_{ij}\,d^4x \qquad (6)$$

Equation of motion for charged particle: $mc\frac{du_i}{ds} = \frac{e}{c}F_{ij}u_j$

$j = 0$ gives: $\frac{d\mathcal{E}}{dt} = e\mathbf{E}\cdot\mathbf{v}$

$j = 1, 2, 3$ gives: $\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)$

Useful variation

$\delta(F^{ij}F_{ij}) = F^{ij}\delta F_{ij} + F_{ij}\delta F^{ij} = 2F^{ij}\delta F_{ij} = 2F^{ij}\delta(\partial_i A_j - \partial_j A_i) = 2F^{ij}\partial_i\delta A_j - 2F^{ij}\partial_j\delta A_i = 4F^{ij}\partial_i\delta A_j = 4\partial_i(F^{ij}\delta A_j) - 4(\delta A_j)\partial_i F^{ij}$

Bianchi identity (p. 88): $\epsilon^{ijkl}\partial_j F_{kl} = 0$

Implications: $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$

Equation of motion for EM field (p. 107): $\partial_i F^{ij} = \frac{4\pi}{c}j^j$

Implications: $\nabla\cdot\mathbf{E} = 4\pi\rho$ and $\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$

Gauge invariance: $\phi' \leftarrow \phi + \frac{1}{c}\frac{\partial\chi}{\partial t}; \mathbf{A}' \leftarrow \mathbf{A} - \nabla\chi$ *i.e.* $A^{i\prime} \leftarrow A^i + \partial^i\chi$ (χ is a Lorentz scalar field)

Properties of Levi–Civita symbol (HW3)

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\lambda} = \delta^{\alpha\beta\gamma}_{\mu\nu\lambda}$$
$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\gamma} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}$$
$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\beta\gamma} = 2\delta_{\alpha\mu}$$
$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\lambda}A_{\alpha\mu}A_{\beta\nu}A_{\gamma\lambda} = 6\det A$$

Useful relations for relativistic mechanics

$$\mathbf{v} = \frac{\mathbf{p}c^2}{\mathcal{E}} \qquad (7)$$
$$\mathbf{p} = \frac{\mathcal{E}\mathbf{v}}{c^2} \qquad (8)$$

Motion in uniform E-field: Integrate equations of motion to obtain energy and momentum, then use (7) to get velocity.

Motion in uniform B-field: Note that energy is constant. Use (8) to write momentum in terms of velocity, then integrate equations of motion to get velocity.

Coulomb gauge: $\phi = 0; \nabla\cdot\mathbf{A} = 0; -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mathbf{0}$

Lorentz gauge: $\partial_i A^i = 0; \partial_i\partial^i A_j = \Box A_j = 0$

Fourier series

$$f(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\tilde f(k_n)e^{ik_n x}$$
$$\tilde f(k_n) = \int_{-L/2}^{L/2} f(x)e^{-ik_n x}\,dx$$
$$k_n = \frac{2\pi n}{L}$$

Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde f(k)e^{ikx}\,dk$$
$$\tilde f(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$
$$\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx = 2\pi\delta(k - k')$$

Need $\tilde f(k) = \tilde f^*(-k)$ in order for $f(x)$ to be real.

Solution of wave equation for EM field in Coulomb gauge (p. 121)

$$\mathbf{A}(\mathbf{x}, t) = \iiint \vec\beta^*(-\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)} + \vec\beta(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,\frac{d^3k}{(2\pi)^3}$$

Monochromatic plane wave

$\vec\beta(\mathbf{k}) = \vec\beta\delta^3(\mathbf{k} - \mathbf{p})(2\pi)^3$ which yields

$\mathbf{A}(\mathbf{x}, t) = 2\vec\beta\cos(\mathbf{p}\cdot\mathbf{x} - \omega t)$ (p. 122)

$\mathbf{k}\cdot(\mathbf{E}\times\mathbf{B}) > 0$ (p. 123)

$\omega = c\|\mathbf{k}\|$ (p. 124)

Poynting vector and EM energy density (p. 126)

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}$$
$$\mathcal{E}_{\text{em}} = \frac{1}{8\pi}(E^2 + B^2)$$
$$\frac{\partial}{\partial t}\left[\frac{1}{8\pi}(E^2 + B^2)\right] = -\mathbf{j}\cdot\mathbf{E} - \nabla\cdot\mathbf{S}$$
$$\frac{d}{dt}\int_V \frac{E^2 + B^2}{8\pi}\,d^3x = -\int_V \mathbf{j}\cdot\mathbf{E}\,d^3x - \int_S \mathbf{S}\cdot d\mathbf{a}$$

Laplace Green function (GF1)

$G(\mathbf{x}) = \frac{1}{4\pi\|\mathbf{x}\|}; \nabla^2 G(\mathbf{x}) = -\delta^3(\mathbf{x})$

d'Alembert Green function (GF5)

$G(x^i) = -\frac{\delta(ct - r)}{4\pi r}; \Box G(x^i) = -\delta^4(x^i)$

Energy of plane wave (p. 128): $\mathbf{S} = \hat{\mathbf{k}}c\mathcal{E}_{\text{em}}$

EM momentum density (p. 129): $\vec{\mathcal{P}}_{\text{em}} = \frac{\mathbf{S}}{c^2}$

Inhomogeneous wave equation in Lorenz gauge (p. 133): $\Box A^i = \frac{4\pi}{c}j^i$

Retarded four-potentials in Lorenz gauge (142.1)

$$A^i(\mathbf{x}, t) = \frac{1}{c}\int \frac{j^i(\mathbf{x}', t - \|\mathbf{x} - \mathbf{x}'\|/c)}{\|\mathbf{x} - \mathbf{x}'\|}\,d^3x'$$

Four-current for point particle (p. 142)

$$\rho(\mathbf{x}, t) = e\delta^3(\mathbf{x} - \mathbf{x}_c(t))$$
$$\mathbf{j}(\mathbf{x}, t) = e\dot{\mathbf{x}}_c(t)\delta^3(\mathbf{x} - \mathbf{x}_c(t))$$

Liénard–Wiechert potentials (p. 146)

$$A^0(\mathbf{x}, t) = \frac{ec}{cr_r - \mathbf{v}_r\cdot\mathbf{r}_r}$$
$$\mathbf{A}(\mathbf{x}, t) = \frac{e\mathbf{v}_r}{cr_r - \mathbf{v}_r\cdot\mathbf{r}_r}$$

where $\mathbf{r}_r = \mathbf{x} - \mathbf{x}_c(t_r)$; $\mathbf{v}_r = \dot{\mathbf{x}}_c(t_r)$; and t_r is the retarded time and satisfies $c(t - t_r) = \|\mathbf{x} - \mathbf{x}_c(t_r)\|$.

E and B fields of point charge (p. 148)

$$\mathbf{u} = c\hat{\mathbf{r}}_r - \mathbf{v}_r$$
$$\mathbf{E} = e\frac{r_r}{(\mathbf{r}_r\cdot\mathbf{u})^3}[(c^2 - v_r^2)\mathbf{u} + \mathbf{r}_r\times(\mathbf{u}\times\ddot{\mathbf{x}}_c(t_r))]$$
$$\mathbf{B} = \hat{\mathbf{r}}_r\times\mathbf{E}$$

Approximate fields of localized charge configuration ($x' \ll r, x' \ll \lambda$) (p. 158)

$$A^0(\mathbf{r}, t) \approx \frac{q}{r} + \frac{\hat{\mathbf{r}}}{r^2}\cdot\mathbf{d}(t_0) + \frac{1}{c}\frac{\hat{\mathbf{r}}}{r}\cdot\dot{\mathbf{d}}(t_0)$$
$$\mathbf{A}(\mathbf{r}, t) \approx \frac{1}{c}\frac{1}{r}\dot{\mathbf{d}}(t_0)$$

where $t_0 = t - r/c$, and $\mathbf{d}$ is the dipole moment.

E and B fields for pure dipole (p. 163), Poynting vector (p. 164), and total power radiated (p. 165)

$$\mathbf{E} = \frac{1}{c^2 r}(\ddot{\mathbf{d}}(t_0)\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}$$
$$\mathbf{B} = \hat{\mathbf{r}}\times\mathbf{E}$$
$$\mathbf{S} = \hat{\mathbf{r}}\sin^2\theta\frac{\|\ddot{\mathbf{d}}(t_0)\|^2}{4\pi c^3 r^2}$$
$$P_{\text{tot}} = \frac{2}{3c^2}\|\ddot{\mathbf{d}}(t_0)\|^2$$

Power of oscillating dipole, $d = qa\cos\omega t$ (p. 165)

$\langle P_{\text{tot}}\rangle = \frac{e^2a^2}{3c^3}\omega^4$

Electromagnetic stress-energy tensor

$$T^{km} = -\frac{1}{4\pi}F^{kj}F^m{}_j + \frac{\eta^{km}}{16\pi}F^{ij}F_{ij}$$
$$\partial_k T^{km} = 0 \quad (172.1)$$
$$T^{km} = T^{mk} \quad (\text{p. } 172)$$
$$T^{00} = \frac{1}{8\pi}(E^2 + B^2) = \mathcal{E}_{\text{em}}$$
$$T^{\alpha 0} = T^{0\alpha} = \frac{\mathbf{S}}{c}$$
$$T^{\alpha\beta} = -\frac{1}{4\pi}\left[E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}(E^2 + B^2)\right] \quad (\text{p. } 176)$$
$$\sigma_{\alpha\beta} = -T^{\alpha\beta}$$

σ is Maxwell stress tensor. $T^{\alpha 0}$ is c times momentum density. $T^{0\alpha}$ is $1/c$ times energy flux density. $T^{\alpha\beta}$ is momentum flux.

Total electromagnetic force on volume (p. 178)

$$F_V^\beta = \int_S n^\alpha T^{\alpha\beta}da$$
$$F_V^\beta = f_V^\beta + \frac{1}{c^2}\frac{d}{dt}\int_V \mathbf{S}\,d^3x$$

n is inward unit normal. f_V^β is total force on charges and currents,

$$f_V^\beta = \int_V \rho\mathbf{E} + \frac{\mathbf{j}}{c}\times\mathbf{B}\,d^3x$$

$\frac{1}{c^2}\int_V \mathbf{S}\,d^3x$ is total electromagnetic momentum.

Classical electron radius (p. 187): $r_e = \frac{e}{m_e c^2} \approx 10^{-13}$ cm

Radiation reaction (p. 198): $m\mathbf{a} = \mathbf{F} + \frac{2e^2}{3c^3}\dot{\mathbf{a}}$

Electromagnetic duality:

$$\mathbf{E} \to \mathbf{B}$$
$$\mathbf{B} \to -\mathbf{E}$$
$$\mathbf{d} \to \frac{\vec\mu}{c}$$
$$\vec\mu \to -c\mathbf{d}$$

$\mathbf{d}$ is electric dipole moment, $\vec\mu$ is magnetic dipole moment

Potential of uniformly moving charge ($\mathbf{x}' = (vt, 0, 0)$, HW2)

$$\phi = \frac{\gamma e}{\sqrt{\gamma^2(x - vt)^2 + y^2 + z^2}}$$

$A_x = \beta\phi \quad A_y = 0 \quad A_z = 0$

Field of uniformly moving charge (HW3)

$$E_x = \frac{\gamma e(x - vt)}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}} \qquad B_x = 0$$
$$E_y = \frac{\gamma ey}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}} \qquad B_y = -\beta E_z$$
$$E_z = \frac{\gamma ez}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}} \qquad B_z = \beta E_y$$

i.e., $\mathbf{B} = \vec\beta\times\mathbf{E}$

Boundary conditions for E and B fields

$$E_\perp^+ - E_\perp^- = 4\pi\sigma \qquad B_\perp^+ - B_\perp^- = 0$$
$$\mathbf{E}_\parallel^+ - \mathbf{E}_\parallel^- = \mathbf{0} \qquad \mathbf{B}_\parallel^+ - \mathbf{B}_\parallel^- = \frac{4\pi}{c}\vec\kappa\times\hat{\mathbf{n}}$$

σ is surface charge density, $\vec\kappa$ is surface current density